

THE ONE-THIRD-TRICK AND SHIFT OPERATORS

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ABSTRACT. In this paper we obtain a representation as martingale transform operators for the rearrangement and shift operators introduced by T. Figiel in [Fig88] and [Fig90]. The martingale transforms and the underlying sigma algebras are obtained explicitly by combinatorial means. The known norm estimates for those operators are a direct consequence of our representation.

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1. INTRODUCTION

The proof of the $T(1)$ theorem by T. Figiel proceeds by expanding the integral operator into an absolutely converging series of basic building blocks T_m and U_m , rearranging and shifting the Haar system. This involves the following norm estimates for those building blocks, which T. Figiel obtained by combinatorial means:

$$\|T_m : L_X^p \rightarrow L_X^p\| \leq C (\log_2(2 + |m|))^\alpha, \quad (1.1)$$

$$\|U_m : L_X^p \rightarrow L_X^p\| \leq C (\log_2(2 + |m|))^\beta, \quad (1.2)$$

where the constant $C > 0$ depends only on p , the UMD-constant of X and $0 < \alpha, \beta < 1$. For the original proof see [Fig88] and [FW01]. See also [NS97] and [Mül05]. For extensions to spaces of the homogeneous type see [MP11].

The purpose of the present paper is to obtain a representation of T_m and U_m as the sum of roughly $\log_2(2 + |m|)$ martingale transform operators. This is done by combinatorial analysis of the equations defining T_m and U_m .

Our combinatorial analysis exhibits the link of T. Figiel's rearrangement and shift operators to the so called one-third-trick originating in the work of [Wol82], [GJ82], [Dav80] and [CWW85].

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Related Recent Developments. This article is taken from my Ph.D. thesis [Lec11].

Recently T. P. Hytönen, see [Hyt11], presented his own proof of T. Figiel's vector-valued $T(1)$ theorem, see [Fig90]. The basic aim of T. P. Hytönen in [Hyt11] is the same as ours, to find reductions of the general case to certain preferable situations where the so called Figiel compatibility condition is satisfied. To this end the problem in [Hyt11] is randomized and the properties of the so called random dyadic partitions of Nazarov, Treil and Volberg, see [NTV97, NTV03] are exploited.

By contrast the proof in the present paper proceeds by finding explicitly those filtrations that turn a given bad combinatorial situation into a good one, such that Figiel's compatibility condition is satisfied. Our reduction is self-contained and develops specific combinatorics of colored dyadic intervals.

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2. PRELIMINARIES

The Haar System in \mathbb{R} .

Consider the collection of dyadic intervals at scale $j \in \mathbb{Z}$ given by

$$\mathcal{D}_j = \{ [2^{-j}k, 2^{-j}(k+1)[: k \in \mathbb{Z} \},$$

and the collection of the dyadic intervals

$$\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j.$$

We define the L^∞ -normalized Haar system by

$$h_{[0,1[}(t) = 1_{[0, \frac{1}{2}[}(t) - 1_{[\frac{1}{2}, 1[}(t), \quad t \in \mathbb{R},$$

and for every $I \in \mathcal{D}$ set

$$h_I(t) = h_{[0,1[}\left(\frac{t - \inf I}{|I|}\right), \quad t \in \mathbb{R},$$

where 1_A denotes the characteristic function of a set A .

Banach Spaces with the UMD-Property.

By $L^p(\Omega, \mu; X)$ we denote the space of functions with values in X , Bochner-integrable with respect to μ . If $\Omega = \mathbb{R}$ and μ is the Lebesgue measure $|\cdot|$ on \mathbb{R} , then set $L_X^p(\mathbb{R}) = L^p(\mathbb{R}, |\cdot|; X)$, if unambiguous further abbreviated as L_X^p .

We say X is a UMD space if for every X -valued martingale difference sequence $\{d_j\}_j \subset L^p(\Omega, \mu; X)$, $1 < p < \infty$, and choice of signs $\varepsilon_j \in \{-1, 1\}$ one has

$$\left\| \sum_j \varepsilon_j d_j \right\|_{L^p(\Omega, \mu; X)} \leq \mathcal{U}_p(X) \cdot \left\| \sum_j d_j \right\|_{L^p(\Omega, \mu; X)}, \quad (2.1)$$

where $\mathcal{U}_p(X)$ does not depend on ε_j or d_j . The constant $\mathcal{U}_p(X)$ is called UMD-constant. We refer the reader to [Bur81].

Kahane's Contraction Principle.

For every Banach space X , $1 \leq p < \infty$, finite set $\{x_j\}_j \subset X$ and bounded sequence of scalars $\{c_j\}_j$ we have

$$\left(\int_0^1 \left\| \sum_j r_j(t) c_j x_j \right\|_X^p dt \right)^{1/p} \leq \sup_j |c_j| \cdot \left(\int_0^1 \left\| \sum_j r_j(t) x_j \right\|_X^p dt \right)^{1/p}, \quad (2.2)$$

where $\{r_j\}_j$ denotes an independent sequence of Rademacher functions. For details see [Kah85]. Below we give a short proof, see [Kah85].

Proof. By scaling inequality (2.2), we may assume $|c_j| \leq 1$, for all j . We represent each c_j as the series $c_j = \sum_{k \geq 1} \varepsilon_{jk} 2^{-k}$, with suitable $\varepsilon_{jk} \in \{\pm 1\}$ and observe

$$\begin{aligned} \left(\int_0^1 \left\| \sum_j r_j(t) c_j x_j \right\|_X^p dt \right)^{1/p} &\leq \sum_{k \geq 1} 2^{-k} \left(\int_0^1 \left\| \sum_j r_j(t) \varepsilon_{jk} x_j \right\|_X^p dt \right)^{1/p} \\ &= \left(\int_0^1 \left\| \sum_j r_j(t) x_j \right\|_X^p dt \right)^{1/p}. \end{aligned}$$

The last equality holds true since $\sum_j r_j(t) \varepsilon_{jk} x_j$ has the same distribution as $\sum_j r_j(t) x_j$ for all choices of signs ε_{jk} . \square

The Martingale Inequality of Stein – Bourgain's Version.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{F}_1 \subset \dots \subset \mathcal{F}_m \subset \mathcal{F}$ denote an increasing sequence of σ -algebras. For every choice of $f_1, \dots, f_m \in L^p(\Omega, \mu; X)$, $1 < p < \infty$, let r_1, \dots, r_m denote independent Rademacher functions, then

$$\int_0^1 \left\| \sum_{i=1}^m r_i(t) \mathbb{E}(f_i | \mathcal{F}_i) \right\|_{L^p(\Omega, \mu; X)} dt \leq C \cdot \int_0^1 \left\| \sum_{i=1}^m r_i(t) f_i \right\|_{L^p(\Omega, \mu; X)} dt, \quad (2.3)$$

where C depends only on p and $\mathcal{U}_p(X)$. The scalar valued version of (2.3) by E. M. Stein can be found in [Ste70]. The vector valued extension is due to J. Bourgain [Bou86]. A proof may be found in [FW01].

Additional Notation.

Let \mathcal{N} be a collection of nested sets, then $\pi_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N}$ is defined as follows. Let $K \in \mathcal{N}$ and then $\pi_{\mathcal{N}}(K)$ is the smallest element with respect to inclusion of the collection $\{M \in \mathcal{N} : K \subsetneq M\}$. In most cases we will omit the subscript and explicitly state to which nested collection we refer.

Given a collection of Lebesgue measurable sets \mathcal{L} , the collection of Lebesgue measurable sets σ -algebra(\mathcal{L}) denotes the smallest sigma algebra containing \mathcal{L} .

3. THE ONE-THIRD-TRICK

We will introduce and investigate two variants of one-third-shift operators, that is the bilateral alternating one-third-shift operator and the unilateral one-third-shift operator. First we will introduce the bilateral alternating one-third-shift operator S given by $S(h_I) = h_{\sigma(I)}$, see (3.4). Roughly speaking, σ shifts intervals, say for example having length 1, to the *right* by one third of their length, so in our instance by $\frac{1}{3}$. The intervals having length $\frac{1}{2}$ are then shifted by one third of their size to the *left*, so by $\frac{1}{6}$. Hence the relative translation of two successive levels of dyadic intervals amounts to a total of $\frac{1}{2}$, thus yielding a nested collection of intervals, again. This is illustrated in Figure 1 on the following page. In Theorem 3.2 we establish that $S : L_X^p \rightarrow L_X^p$ is an isomorphism by means of Bourgain's version of Stein's martingale inequality.

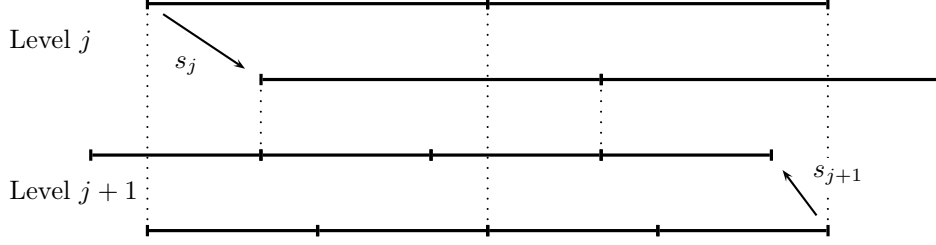


FIGURE 1. One-third-shift of two consecutive levels of intervals.
In this illustration j is even.

Finally, we will consider the unilateral variants S_0 and S_1 of the one-third-shift operator, and establish in Theorem 3.3 that both are isomorphic maps from L_X^p to itself, as well.

The one-third-trick originates with the work of [Wol82], [GJ82] and [CWW85].

3.1. Bilateral Alternating One-Third-Shift.

For every $j \in \mathbb{Z}$ let

$$s_j = (-1)^j 2^{-j}/3, \quad (3.1)$$

and define

$$s(I) = s_j, \quad (3.2)$$

for all intervals I having measure $|I| = 2^{-j}$. Then define the one-third-shift map

$$\sigma(I) = I + s(I), \quad (3.3)$$

and the one-third-shift operator

$$S(h_I) = h_{\sigma(I)}, \quad (3.4)$$

where by $h_{\sigma(I)}$ we denote the function $h_{\sigma(I)}(x) = h_I(x - s(I))$. The one-third-shift of dyadic intervals for two consecutive levels is illustrated in Figure 1.

From this picture one can see that the collection of one-third-shifted dyadic intervals $\sigma(\mathcal{D})$ is nested, and $\mathcal{D} \cap \sigma(\mathcal{D}) = \emptyset$. Note that if a one-third-shifted dyadic interval $J \in \sigma(\mathcal{D})$ is contained in a non-shifted interval $I \in \mathcal{D}$, then $\text{dist}(J, I^c) \geq |J|/3$. For every given interval $I \in \mathcal{D}$ exists a unique one-third-shifted interval $J \in \sigma(\mathcal{D})$, $|J| = |I|/2$ being contained in I . First observe that for every $j \in \mathbb{Z}$ and $I \in \mathcal{D}_j$ we have

$$\begin{aligned} \#\{J \in \sigma(\mathcal{D}_{j+1}) : J \cap I \neq \emptyset\} &= 3, \\ \#\{J \in \sigma(\mathcal{D}_{j+1}) : J \subset I\} &= 1. \end{aligned}$$

So we can define $\omega(I)$ by

$$\omega(I) = J, \quad \text{where } J \in \sigma(\mathcal{D}), |J| = |I|/2 \text{ and } J \subset I, \quad (3.5)$$

see Figure 2 on the facing page.

Note the basic properties summarized in

Lemma 3.1. *The following statements are true.*

- (i) $\sigma(\mathcal{D})$ is a nested collection of dyadic intervals, and $\mathcal{D} \cap \sigma(\mathcal{D}) = \emptyset$.
- (ii) $\omega : \mathcal{D} \rightarrow \sigma(\mathcal{D})$ is well defined and injective.
- (iii) Let $I \in \mathcal{D}$, then $\omega(I) \subset I$.
- (iv) For every $I \in \mathcal{D}$ we have $\text{dist}(\omega(I), I^c) = |I|/6$.

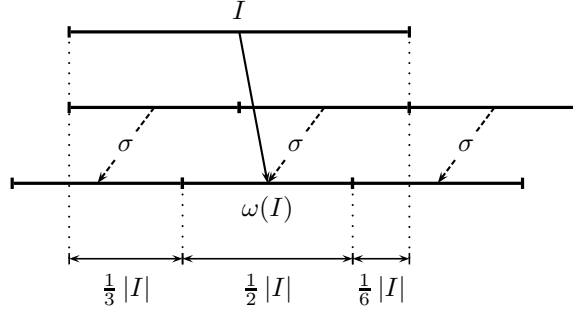


FIGURE 2. The interval I has measure $|I| = 2^{-j}$ with j being even.

- (v) Let $I, J \in \mathcal{D}$, $|I| = |J|$, then $\text{dist}(\omega(I), \omega(J)) < |\omega(I)|$ if and only if $I = J$.
 (vi) For all $I \in \mathcal{D}$ we have the identity $\sigma(I) = \omega(I) \cup (\omega(I) + \text{sign}(s(I)) \cdot |\omega(I)|)$.

Proof. The assertions are easily verified. \square

We need to build up some more notation. For all $j \in \mathbb{Z}$ and

$$u = \sum_{I \in \mathcal{D}} u_I h_I |I|^{-1}$$

let $(u)_j$ restrict the function u to level j , precisely

$$(u)_j = \sum_{I \in \mathcal{D}_j} u_I h_I |I|^{-1}. \quad (3.6)$$

If we define

$$\mathbb{I}(u)_j = \sum_{I \in \mathcal{D}_j} u_I \mathbf{1}_I |I|^{-1}, \quad (3.7)$$

then we find due to Kahane's contraction principle (2.2) that

$$\int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (u)_j \right\|_{L_X^p} dt = \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(u)_j \right\|_{L_X^p} dt. \quad (3.8)$$

The following theorem establishes that the one-third-shift operator $S : L_X^p \rightarrow L_X^p$ is an isomorphism.

Theorem 3.2. *Let $1 < p < \infty$ and X a Banach space with the UMD-property, then there exists a constant $C > 0$ such that*

$$\frac{1}{C} \|u\|_{L_X^p} \leq \|Su\|_{L_X^p} \leq C \|u\|_{L_X^p},$$

for all $u \in L_X^p$. The constant C depends only on $\mathcal{U}_p(X)$.

Proof. Let $u = \sum_{I \in \mathcal{D}} u_I h_I |I|^{-1} \in L_X^p$ be fixed throughout this proof and set

$$v = \sum_{I \in \mathcal{D}} u_I h_{\omega(I)} |\omega(I)|^{-1}.$$

Note that $\{\omega(I) : I \in \mathcal{D}\}$ is nested, see Lemma 3.1, assertion (i) and (ii). Observe we have due to Lemma 3.1, assertion (iii) that $\mathbb{I}(u)_j = \mathbb{E}(\mathbb{I}(v)_j | \mathcal{D}_j)$, so the UMD-property and Kahane's contraction principle (2.2) yield

$$\begin{aligned} \|u\|_{L_X^p} &\lesssim \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (u)_j \right\|_{L_X^p} dt \\ &= \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(u)_j \right\|_{L_X^p} dt \\ &= \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{E}(\mathbb{I}(v)_j | \mathcal{D}_j) \right\|_{L_X^p} dt. \end{aligned}$$

Now we apply Stein's martingale inequality (2.3) followed by identity (3.8) to pass from $\mathbb{I}(v)_j$ to $(v)_j$, so

$$\begin{aligned} \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{E}(\mathbb{I}(v)_j | \mathcal{D}_j) \right\|_{L_X^p} dt &\lesssim \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(v)_j \right\|_{L_X^p} dt \\ &= \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (v)_j \right\|_{L_X^p} dt. \end{aligned}$$

Recalling definition (3.4) and applying Kahane's contraction principle in consideration of $\omega(I) \subset \sigma(I)$ (see identity (vi) in Lemma 3.1), we estimate

$$\int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (v)_j \right\|_{L_X^p} dt \leq 2 \cdot \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (Su)_j \right\|_{L_X^p} dt,$$

and the UMD-property implies

$$\int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) (v)_j \right\|_{L_X^p} dt \lesssim \|Su\|_{L_X^p}.$$

Thus, collecting the inequalities yields

$$\|u\|_{L_X^p} \lesssim \|Su\|_{L_X^p}.$$

One can repeat the preceding argument with the roles of u and Su interchanged and obtain the converse inequality

$$\|Su\|_{L_X^p} \lesssim \|u\|_{L_X^p}.$$

□

3.2. Unilateral One-Third-Shift.

We now introduce modified versions σ_0 and σ_1 of the one-third-shift map σ . To this end we define $\sigma_0, \sigma_1 : \mathcal{D} \rightarrow \sigma(\mathcal{D})$,

$$\sigma_0(I) = J, \quad \text{where } J \in \sigma(\mathcal{D}), |J| = |I| \text{ and } \sup J \in I, \quad (3.9)$$

$$\sigma_1(I) = J, \quad \text{where } J \in \sigma(\mathcal{D}), |J| = |I| \text{ and } \inf J \in I, \quad (3.10)$$

see Figure 3 on the next page. This induces the one-third-shift operators S_0 and S_1 given by the linear extension of

$$S_0(h_I) = h_{\sigma_0(I)}, \quad I \in \mathcal{D}, \quad (3.11)$$

$$S_1(h_I) = h_{\sigma_1(I)}, \quad I \in \mathcal{D}. \quad (3.12)$$

Observe that we have either

$$\sigma(I) = \sigma_0(I) \quad \text{or} \quad \sigma(I) = \sigma_1(I),$$

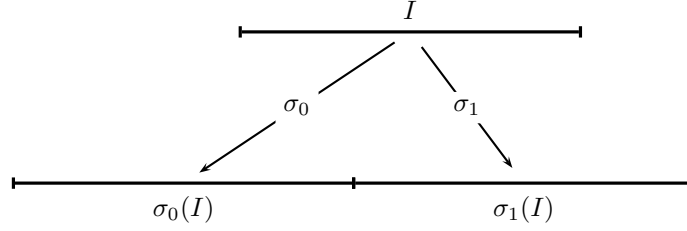


FIGURE 3. Unilateral one-third-shifts σ_0 and σ_1 applied to $I \in \mathcal{D}$. In this picture the one-third-shift map σ shifts to the right, so $\sigma_1(I) = \sigma(I)$.

depending on the direction in which σ one-third-shifts the interval I . Anyhow, we can see that

$$|I \cap \sigma_0(I)| \geq \frac{1}{3} |I|, \quad |I \cap \sigma_1(I)| \geq \frac{1}{3} |I|,$$

for all $I \in \mathcal{D}$. The proof of Theorem 3.2 on page 5 with minor modifications yields Theorem 3.3 below.

Theorem 3.3. *Let $1 < p < \infty$ and X a Banach space with the UMD-property, then there exists a constant $C > 0$ such that*

$$\begin{aligned} \frac{1}{C} \|u\|_{L_X^p} &\leq \|S_0 u\|_{L_X^p} \leq C \|u\|_{L_X^p}, \\ \frac{1}{C} \|u\|_{L_X^p} &\leq \|S_1 u\|_{L_X^p} \leq C \|u\|_{L_X^p}, \end{aligned}$$

for all $u \in L_X^p$. The constant C depends only on $\mathcal{U}_p(X)$.

Proof. Define ω_0 and ω_1 by

$$\begin{aligned} \omega_0(I) &= J, & \text{where } J \in \sigma(\mathcal{D}), |J| &= |I|/4 \text{ and } \sup J = \sup \sigma_0(I), \\ \omega_1(I) &= J, & \text{where } J \in \sigma(\mathcal{D}), |J| &= |I|/4 \text{ and } \inf J = \inf \sigma_1(I), \end{aligned}$$

for all $I \in \mathcal{D}$. Now all we need to do is repeat the proof of Theorem 3.3 with ω replaced by ω_δ in order to estimate S_δ , for each $\delta \in \{0, 1\}$. \square

4. THE SHIFT OPERATOR T_m

Here we define $16 + 4 \cdot \log_2(|m|)$, $m \neq 0$ collections of the Haar system, so that on each such subcollection T_m acts as a martingale transform operator on either the dyadic grid or the one-third-shifted dyadic grid. In section 3 we established that changing the dyadic grid to the one-third-shifted dyadic grid is an isomorphism. Thus we may assume that T_m is representable as a martingale transform operator on each of the $16 + 4 \cdot \log_2(|m|)$ subcollections, which yields the well known estimate

$$\|T_m : L_X^p \rightarrow L_X^p\| \leq C \cdot (\log_2(2 + |m|))^\alpha,$$

for some $0 < \alpha < 1$, established by T. Figiel in [Fig88].

Define the shift map τ_m , $m \in \mathbb{Z}$ by

$$\tau_m(I) = I + m|I|, \tag{4.1}$$

for all $I \in \mathcal{D} \cup \sigma(\mathcal{D})$. This induces the shift operator T_m , given by

$$T_m h_I = h_{\tau_m(I)}, \tag{4.2}$$

for all $I \in \mathcal{D} \cup \sigma(\mathcal{D})$. It is crucial that the one-third-shift operator S defined in (3.4) and the shift operator T_m commute, that is the identity

$$(S \circ T_m)(u) = (T_m \circ S)(u), \quad (4.3)$$

for all $u \in L_X^p$. Analogously, we have that

$$(S_0 \circ T_m)(u) = (T_m \circ S_0)(u), \quad (4.4)$$

$$(S_1 \circ T_m)(u) = (T_m \circ S_1)(u), \quad (4.5)$$

for all $u \in L_X^p$, see (3.9), (3.10), (3.11) and (3.12).

We aim at splitting the dyadic intervals \mathcal{D} into collections $\mathcal{B}_i^{(\delta)}$, such that we may bound $T_m \circ S^\delta$ on functions supported on $\sigma^\delta(\mathcal{B}_i^{(\delta)})$, $\delta \in \{0, 1\}$. Note that if $\delta = 0$, then $S^\delta = \text{Id}$ and $\sigma^\delta = \text{Id}$.

Given a shift width $m \in \mathbb{Z}$, $m \neq 0$, we will partition the dyadic intervals \mathcal{D} into $16 + 4 \cdot \log_2(|m|)$ disjoint collections denoted by $\mathcal{B}_i^{(\delta)}$. The collections are constructed in such way that for each i and $\delta \in \{0, 1\}$ fixed, we have that whenever $I \in \mathcal{B}_i^{(\delta)}$, the intervals $\sigma^\delta(I)$ and $(\tau_m \circ \sigma^\delta)(I)$ share the same dyadic predecessor with respect to the collection $\sigma^\delta(\mathcal{B}_i^{(\delta)})$. The details are elaborated in Lemma 4.1 below.

Lemma 4.1. *For every integer $m \in \mathbb{Z}$, $m \neq 0$ let τ_m denote the map given by*

$$\tau_m(I) = I + m|I|,$$

for all $I \in \mathcal{D} \cup \sigma(\mathcal{D})$, see (4.1).

Then there exist a constant $K(m) \leq 7 + 2 \cdot \log_2(|m|)$ and disjoint collections of dyadic intervals $\mathcal{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ with

$$\mathcal{D} = \bigcup_{\delta \in \{0, 1\}} \bigcup_{i=0}^{K(m)} \mathcal{B}_i^{(\delta)},$$

such that

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \sigma^\delta(\mathcal{B}_i^{(\delta)})\} \quad (4.6)$$

is a nested collection of sets, for all $0 \leq i \leq K(m)$ and $\delta \in \{0, 1\}$.

Proof. Due to symmetry we may assume that $m \geq 1$, and we set $K(m) = K(-m)$, if $m \leq -1$. So fix a shift width $m \geq 2$ and a $\lambda \geq 4$ such that

$$2^{\lambda-3} \leq m < 2^{\lambda-2}, \quad (4.7)$$

and define $L(m) = \lambda - 1$. If $m = 1$, then let $\lambda = 4$ and set $L(1) = 3$. Now we split \mathcal{D} into disjoint collections \mathcal{A}_i , $0 \leq i \leq L(m)$, by omitting $L(m)$ consecutive levels of \mathcal{D} . More precisely, for every $0 \leq i \leq L(m)$ we define

$$\mathcal{A}_i = \bigcup_{j \in \mathbb{Z}} \{I \in \mathcal{D} : |I| = 2^{-(\lambda \cdot j + i)}\}. \quad (4.8)$$

Next we want to divide each of the \mathcal{A}_i into two collections $\mathcal{A}_i^{(0)}$ and $\mathcal{A}_i^{(1)}$, such that every $I \in \mathcal{A}_i^{(0)}$ has the same predecessor in $\mathcal{A}_i^{(0)}$ as $\tau_m(I)$, and $\mathcal{A}_i^{(0)}$ is maximal. As a consequence, the collection $\mathcal{A}_i^{(1)}$ consists all intervals I such that I and $\tau_m(I)$ do not share the same predecessor. But, if we apply the one-third-shift map σ to the collection $\mathcal{A}_i^{(1)}$, then every $I \in \sigma(\mathcal{A}_i^{(1)})$ has the same predecessor in

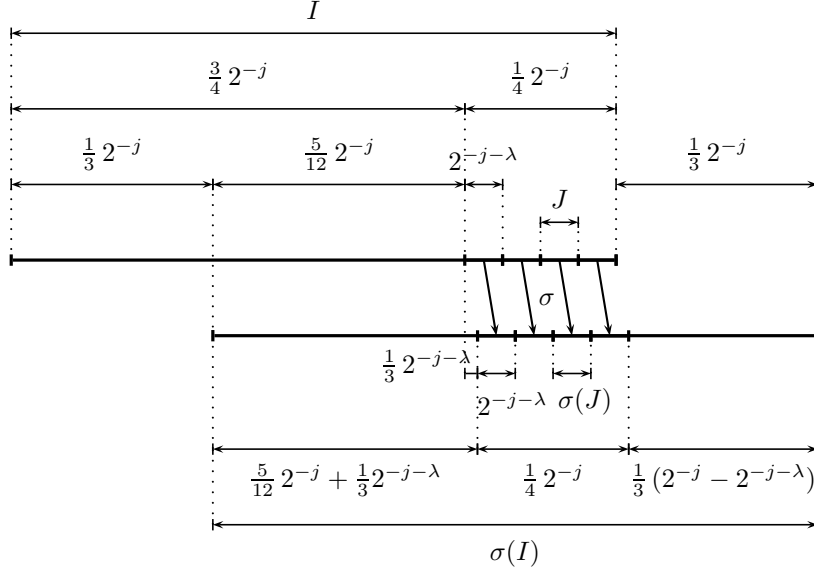


FIGURE 4. The one-third-shift map σ acting on $I \in \mathcal{D}$, $|I| = 2^{-j}$ and $J \in \mathcal{D}$, $|J| = 2^{-j-\lambda}$, where $J \subset I$ and $\tau_m(J) \cap I = \emptyset$. In this picture λ is even.

$\sigma(\mathcal{A}_i^{(1)})$ as $\tau_m(I)$. We will now construct these two collections. To this end let \mathcal{G} denote one of the collections $\mathcal{A}_i, \sigma(\mathcal{A}_i)$, $0 \leq i \leq L(m)$ and define

$$\begin{aligned} \mathcal{C}_0(\mathcal{G}, I) &= \{J \in \mathcal{G} : |J| = 2^{-\lambda} |I|, J \subset I \text{ and } \tau_m(J) \subset I\}, \\ \mathcal{C}_1(\mathcal{G}, I) &= \{J \in \mathcal{G} : |J| = 2^{-\lambda} |I|, J \subset I \text{ and } \tau_m(J) \cap I = \emptyset\}. \end{aligned} \quad (4.9)$$

Revisiting the definition of the one-third-shift map (3.3) and considering the restriction (4.7) one can see that

$$\sigma(\mathcal{C}_1(\mathcal{A}_i, I)) \subset \mathcal{C}_0(\sigma(\mathcal{A}_i), \sigma(I)), \quad (4.10)$$

for all $I \in \mathcal{A}_i$, $0 \leq i \leq L(m)$. This means that all intervals $J \in \sigma(\mathcal{C}_1(\mathcal{A}_i, I))$ are such that J and $\tau_m(J)$ share $\sigma(I)$ as common predecessor with respect to the collection $\sigma(\mathcal{A}_i^{(1)})$. In Figure 4 one can see the action of the one-third-shift map σ on the collection \mathcal{A}_i . Now define for every $0 \leq i \leq L(m)$ the following collections of dyadic intervals

$$\begin{aligned} \mathcal{A}_i^{(0)} &= \bigcup \{\mathcal{C}_0(\mathcal{A}_i, I) : I \in \mathcal{A}_i\}, \\ \mathcal{A}_i^{(1)} &= \mathcal{A}_i \setminus \mathcal{A}_i^{(0)}. \end{aligned} \quad (4.11)$$

Finally, for all $0 \leq i \leq L(m)$ and $\delta \in \{0, 1\}$ we split $\mathcal{A}_i^{(\delta)}$ into two disjoint collections

$$\mathcal{B}_i^{(\delta)} \quad \text{and} \quad \mathcal{B}_{i+L(m)+1}^{(\delta)}, \quad (4.12)$$

such that

$$\mathcal{B}_i^{(\delta)} \cap \tau_m(\mathcal{B}_i^{(\delta)}) = \emptyset, \quad (4.13)$$

for all $0 \leq i \leq K(m)$ and $\delta \in \{0, 1\}$, where we set $K(m) = 2 \cdot L(m) + 1$. Considering (4.7) and $L(m) = \lambda - 1$ we find that $K(m) \leq 7 + 2 \cdot \log_2(m)$. For this purpose consider the collection

$$\mathcal{E} = \{\tau_k(I) : I \in \mathcal{D}, \inf I = 0, 0 \leq k \leq m-1\},$$

and observe that

$$\mathcal{D} = \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ even}}} \tau_{j \cdot m}(\mathcal{E}) \cup \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} \tau_{j \cdot m}(\mathcal{E}) = \mathcal{D}_{\text{even}} \cup \mathcal{D}_{\text{odd}}.$$

Now define the collections

$$\begin{aligned} \mathcal{B}_i^{(\delta)} &= \mathcal{A}_i^{(\delta)} \cap \mathcal{D}_{\text{even}}, \\ \mathcal{B}_{i+L(m)+1}^{(\delta)} &= \mathcal{A}_i^{(\delta)} \cap \mathcal{D}_{\text{odd}}, \end{aligned} \tag{4.14}$$

for all $0 \leq i \leq L(m)$ and $\delta \in \{0, 1\}$.

With regard to (4.10), (4.9) and noting that $\tau_m(I) \in \mathcal{D}_{\text{odd}}$ if and only if $I \in \mathcal{D}_{\text{even}}$, we verified (4.6), finishing this proof. \square

Remark 4.2. Note that we actually proved the slightly stronger result

$$I \cup \tau_m(I) \subset \pi^\lambda(I), \tag{4.15}$$

for all $I \in \sigma^\delta(\mathcal{B}_i^{(\delta)})$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$. Conceive the predecessor map π with respect to $\sigma^\delta(\mathcal{D})$. To be more precise let $I \in \sigma^\delta(\mathcal{D})$. Then $\pi(I)$ is the unique interval $J \in \sigma^\delta(\mathcal{D})$ such that $J \supset I$, and $\pi^\lambda = \pi \circ \dots \circ \pi$.

As the combinatorial Lemma 4.1 on page 8 exhibits the link between the shift map τ_m , the one-third-shift map σ and Figiel's compatibility condition (4.6), the subsequent Theorem 4.3 will translate the combinatorial results into analytical results, exhibiting the link between the shift operator T_m , the one-third-shift operator S and martingale transform operators.

In the following context understand that $1 < p < \infty$, X is a Banach space with the UMD-property and $m \in \mathbb{Z}$, $m \neq 0$. Now we define the projections $P_i^{(\delta)} : L_X^p \longrightarrow L_X^p$, associated with the collections $\mathcal{B}_i^{(\delta)}$ in Lemma 4.1 on page 8

$$P_i^{(\delta)} u = \sum_{I \in \mathcal{B}_i^{(\delta)}} \langle u, h_I \rangle h_I |I|^{-1}, \tag{4.16}$$

for all $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ and $u \in L_X^p$. The Banach space X having the UMD-property implies uniform bounds on the projections $P_i^{(\delta)}$. Note the identity

$$u = \sum_{\delta \in \{0, 1\}} \sum_{i=0}^{K(m)} P_i^{(\delta)} u \tag{4.17}$$

holds true for all $u \in L_X^p$, since the collections $\mathcal{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ form a partition of \mathcal{D} , see Lemma 4.1.

Exploiting that the one-third-shift operator S is an isomorphism on L_X^p (see Theorem 3.2), we will now estimate the shift operator T_m on the range of each $P_i^{(\delta)}$ in the subsequent theorem.

Theorem 4.3. *Let $1 < p < \infty$ and X be a Banach space with the UMD-property. Then for every $m \in \mathbb{Z}$, $0 \leq i \leq K(m)$ and $\delta \in \{0, 1\}$ the inequality*

$$\|T_m \circ P_i^{(\delta)} u\|_{L_X^p} \leq C \cdot \|P_i^{(\delta)} u\|_{L_X^p}, \tag{4.18}$$

holds true for all $u \in L_X^p$, where the constant C depends only on $\mathcal{U}_p(X)$. The projections $P_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ are defined according to (4.16), and $K(m) \leq 7 + 2 \cdot \log_2(1 + |m|)$.

Proof. Note that due to symmetry once we established (4.18) for $m \geq 1$, the theorem is proved.

Recalling the properties of the partition $\mathcal{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ of \mathcal{D} , see Lemma 4.1 on page 8, and we know that the collection

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \sigma^\delta(\mathcal{B}_i^{(\delta)})\} \quad (4.19)$$

is nested, for all $0 \leq i \leq K(m)$ and $\delta \in \{0, 1\}$. Throughout this proof let $m \in \mathbb{Z}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$ and $u \in P_i^{(\delta)}(L_X^p)$ be fixed. According to (4.16) we may assume that u has the representation

$$u = \sum_{I \in \mathcal{B}_i^{(\delta)}} u_I h_I |I|^{-1}.$$

For every $J \in \sigma^\delta(\mathcal{D})$ let

$$A^{(\delta)}(J) = J \cup \tau_m(J), \quad (4.20)$$

and for all $j \in \mathbb{Z}$ define the collection

$$\mathcal{A}_j^{(\delta)} = \{A^{(\delta)}(J) : J \in \sigma^\delta(\mathcal{D}_j)\}. \quad (4.21)$$

Then specify the filtration $\{\mathcal{F}_j^{(\delta)}\}_j$ by

$$\mathcal{F}_j^{(\delta)} = \sigma\text{-algebra}\left(\bigcup_{i \leq j} \mathcal{A}_i^{(\delta)}\right), \quad (4.22)$$

and observe that due to (4.19) every $A^{(\delta)}(J)$, $J \in \sigma^\delta(\mathcal{D}_j)$ is an atom for $\mathcal{F}_j^{(\delta)}$. The one-third-shift operator is given by

$$S^\delta u = \sum_{I \in \mathcal{B}_i^{(\delta)}} u_I h_{\sigma^\delta(I)} |I|^{-1} = \sum_{J \in \sigma^\delta(\mathcal{B}_i^{(\delta)})} u_{\sigma^{-\delta}(J)} h_J |J|^{-1}, \quad (4.23)$$

see (3.4) for details. We recall the notation

$$(u)_j = \sum_{|I|=2^{-j}} u_I h_I |I|^{-1} \quad \text{and} \quad \mathbb{I}(u)_j = \sum_{|I|=2^{-j}} u_I \mathbf{1}_I |I|^{-1},$$

and note that

$$\|T_m S^\delta u\|_{L_X^p} \approx \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(T_m S^\delta u)_j \right\|_{L_X^p} dt,$$

see (3.6), (3.7) and (3.8). Obviously, $\mathbb{I}(T_m S^\delta u)_j \leq 2 \cdot \mathbb{E}(\mathbb{I}(S^\delta u)_j | \mathcal{F}_j^{(\delta)})$, hence Kahane's contraction principle and Bourgain's version of Stein's martingale inequality yield

$$\begin{aligned} \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(T_m S^\delta u)_j \right\|_{L_X^p} dt &\leq \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) 2 \cdot \mathbb{E}(\mathbb{I}(S^\delta u)_j | \mathcal{F}_j^{(\delta)}) \right\|_{L_X^p} dt \\ &\lesssim \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) \mathbb{I}(S^\delta u)_j \right\|_{L_X^p} dt \\ &\approx \|S^\delta u\|_{L_X^p}. \end{aligned}$$

Combining the latter two estimates with Theorem 3.2 on page 5 proves

$$\|T_m S^\delta u\|_{L_X^p} \lesssim \|u\|_{L_X^p}. \quad (4.24)$$

According to (4.3) the shift operator T_m and the one-third-shift operator S commute, so we have the identity

$$T_m u = (S^{-\delta} \circ T_m \circ S^\delta)(u),$$

and we obtain by an application of Theorem 3.2 on page 5

$$\|T_m u\|_{L_X^p} \lesssim \|(T_m \circ S^\delta)(u)\|_{L_X^p}. \quad (4.25)$$

We conclude the proof by joining (4.25) and (4.24). \square

Remark 4.4. By slightly adjusting the construction of $\mathcal{B}_i^{(\delta)}$ we could replace Bourgain's version of Stein's martingale inequality by the martingale transforms in [Fig88, Proposition 2, Step 0] in order to obtain (4.24). To this end we will basically have to replace λ by $\lambda + 1$ and redefine \mathcal{C}_0 and \mathcal{C}_1 as follows

$$\begin{aligned} \mathcal{C}_0(I, \mathcal{A}_i) &= \{J \in \mathcal{A}_i : |J| = 2^{-\lambda} |I|, J \subset I_0 \text{ and } \tau_m(J) \subset I_0\} \\ &\quad \cup \{J \in \mathcal{A}_i : |J| = 2^{-\lambda} |I|, J \subset I_1 \text{ and } \tau_m(J) \subset I_1\}, \\ \mathcal{C}_1(I, \mathcal{A}_i) &= \{J \in \mathcal{A}_i : |J| = 2^{-\lambda} |I|, J \subset I_0 \text{ and } \tau_m(J) \cap I_0 = \emptyset\} \\ &\quad \cup \{J \in \mathcal{A}_i : |J| = 2^{-\lambda} |I|, J \subset I_1 \text{ and } \tau_m(J) \cap I_1 = \emptyset\}, \end{aligned}$$

confer (4.8) and (4.9). This results in the collection

$$\{J_0, \tau_m(J)_0, J_1, \tau_m(J)_1, J \cup \tau_m(J) : J \in \sigma^\delta(\mathcal{B}_i^{(\delta)})\} \quad (4.26)$$

being nested for all $0 \leq i \leq K(m)$ and $\delta \in \{0, 1\}$. With this modifications let us define

$$d_{J,1}^{(\delta)} = \frac{1}{2}(h_J + h_{\tau_m(J)}) \quad \text{and} \quad d_{J,2}^{(\delta)} = \frac{1}{2}(h_J - h_{\tau_m(J)}),$$

for all $J \in \sigma^\delta(\mathcal{B}_i^{(\delta)})$. Since (4.26) is nested, $\{d_{J,1}^{(\delta)}, d_{J,2}^{(\delta)} : J \in \sigma(\mathcal{B}_i^{(\delta)})\}$ forms a martingale difference sequence. Observe $h_J = d_{J,1}^{(\delta)} + d_{J,2}^{(\delta)}$ and $h_{\tau_m(J)} = d_{J,1}^{(\delta)} - d_{J,2}^{(\delta)}$, hence we may swap h_J and $h_{\tau_m(J)}$ without using Bourgain's version of Stein's martingale inequality.

5. A MARTINGALE DECOMPOSITION FOR U_m

In this section we will decompose the Haar system into $24 + 6 \cdot \log_2(|m|)$ subcollections, so that on each fixed subcollection the rearrangement operator U_m is either a martingale transform operator itself or the sum of two martingale transform operators. To be more precise, the total amount of subcollections on which we will estimate parts of U_m that act as martingale transform operators will be $40 + 10 \cdot \log_2(|m|)$. This gives immediately the estimate [Fig88]

$$\|U_m : L_X^p \rightarrow L_X^p\| \leq C \cdot (\log_2(2 + |m|))^\beta,$$

for some $0 < \beta < 1$.

The operator T_m is easier to analyze than U_m . This is mainly due to the observation that $\{T_m h_I\}_{I \in \mathcal{A}}$ is a martingale difference sequence for any choice of $\mathcal{A} \subset \mathcal{D}$, whereas whether $\{U_m h_I\}_{I \in \mathcal{B}}$ forms a martingale difference sequence strongly depends on the choice of $\mathcal{B} \subset \mathcal{D}$. Making use of the one-third-shift operators introduced in Section 3, we will decompose the operator U_m into the five parts

$$U_m = U_m \circ P^{(0)} + \sum_{\varepsilon \in \{0,1\}} (A_m^{(\varepsilon)} + B_m^{(\varepsilon)}) \circ P^{(1,\varepsilon)}$$

each of which behaves like T_m . Some parts of this decomposition will be well localized, whereas others are widespread, see Figures 5, 6 and 7.

In (4.1) we defined the shift map τ_m for every $m \in \mathbb{Z}$ by

$$\tau_m(I) = I + m |I|,$$

for all $I \in \mathcal{D} \cup \sigma(\mathcal{D})$. Now we introduce the shift operator U_m by setting

$$U_m h_I = \mathbf{1}_{\tau_m(I)} - \mathbf{1}_I, \quad (5.1)$$

for all $I \in \mathcal{D} \cup \sigma(\mathcal{D})$. Essentially the same method we used to bound T_m for functions supported on the collections $\mathcal{B}_i^{(0)}$, $0 \leq i \leq K(m)$ qualifies for estimating U_m . This is primarily due to the fact that $\{U_m h_I : I \in \mathcal{B}_i^{(0)}\}$ forms a martingale difference sequence, which is ensured by Lemma 4.1. The main obstacle is to estimate U_m on $\mathcal{B}_i^{(1)}$, since $\{U_m h_I : I \in \mathcal{B}_i^{(1)}\}$ is *not* a martingale difference sequence. The remedy to this problem is the martingale difference sequence decomposition of U_m into

$$U_m h_I = a_I^{(\varepsilon)} + b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}, \quad I \in \mathcal{B}_i^{(1, \varepsilon)}$$

where

$$\begin{aligned} \mathcal{B}_i^{(1,0)} &= \{I \in \mathcal{B}_i^{(1)} : \inf \tau_m(I) \neq \inf \pi^\lambda(\tau_m(I))\}, \\ \mathcal{B}_i^{(1,1)} &= \{I \in \mathcal{B}_i^{(1)} : \inf \tau_m(I) = \inf \pi^\lambda(\tau_m(I))\}. \end{aligned}$$

Recall that given $\delta \in \{0, 1\}$ and an interval $I \in \sigma^\delta(\mathcal{D})$, the interval $\pi(I)$ is the unique $J \in \sigma^\delta(\mathcal{D})$ such that $J \supset I$, and $\pi^\lambda = \pi \circ \dots \circ \pi$. The collections $\{a_I^{(\varepsilon)} : I \in \mathcal{B}_i^{(1, \varepsilon)}\}$ and $\{b_I^{(\varepsilon)}, b_{\tau_m(I)}^{(\varepsilon)} : I \in \mathcal{B}_i^{(1, \varepsilon)}\}$ are martingale difference sequences, each, see Theorem 5.1. This is what enables us to treat U_m like T_m , which is elaborated in Theorem 5.4.

First, we define $\alpha_0, \alpha_1 : \mathcal{D} \longrightarrow \sigma(\mathcal{D})$,

$$\alpha_0(I) = \sigma_0(I), \quad (5.2)$$

$$\alpha_1(I) = \sigma_1(I), \quad (5.3)$$

where σ_0 , and σ_1 are given by (3.9) and (3.10) in Subsection 3.2. Secondly, define the maps β_0, β_1 and β by

$$\beta_0(I) = \alpha_0(I) \setminus I, \quad (5.4)$$

$$\beta_1(I) = \alpha_1(I) \cap I, \quad (5.5)$$

$$\beta(I) = \beta_0(I) \cup \beta_1(I). \quad (5.6)$$

Finally, γ_0, γ_1 and γ are given by

$$\gamma_0(I) = \tau_{-1}(I), \quad (5.7)$$

$$\gamma_1(I) = I, \quad (5.8)$$

$$\gamma(I) = \gamma_0(I) \cup \gamma_1(I). \quad (5.9)$$

The functions $\alpha_0, \alpha_1, \beta_0$ and β_1 are visualized in Figure 5 on the following page.

With $m \in \mathbb{Z}$, $m \geq 1$ fixed, we introduce the functions

$$a_I^{(0)} = \mathbf{1}_{\alpha_0(\tau_m(I))} - \mathbf{1}_{\alpha_0(I)}, \quad I \in \mathcal{D}, \quad (5.10)$$

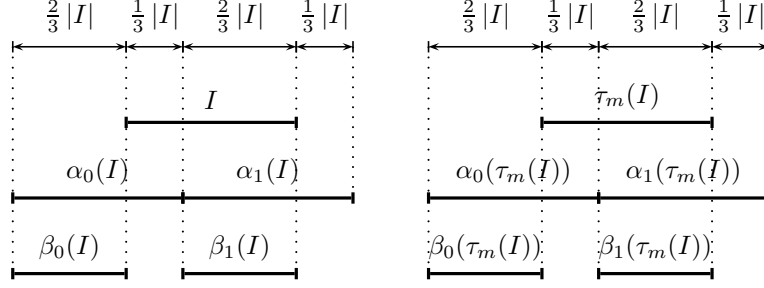
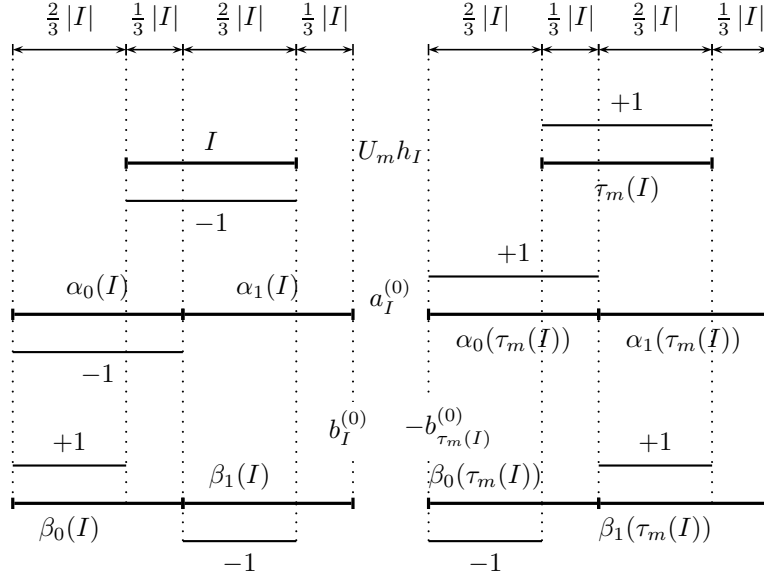
$$b_I^{(0)} = \mathbf{1}_{\beta_0(I)} - \mathbf{1}_{\beta_1(I)}, \quad I \in \mathcal{D}, \quad (5.11)$$

and

$$a_I^{(1)} = \mathbf{1}_{\alpha_1(\tau_m(I))} - \mathbf{1}_{\alpha_1(I)}, \quad I \in \mathcal{D}, \quad (5.12)$$

$$b_I^{(1)} = \mathbf{1}_{I \setminus \beta_1(I)} - \mathbf{1}_{I \setminus \beta_0(I)}, \quad I \in \mathcal{D}. \quad (5.13)$$

see Figures 6 and 7. We define the operators $A_m^{(\varepsilon)}$, $B^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$ as the linear

FIGURE 5. The support functions $\alpha_0, \alpha_1, \beta_0, \beta_1$ for I and $\tau_m(I)$.FIGURE 6. Martingale decomposition of U_m to the left.

extension of

$$A_m^{(\varepsilon)} h_I = a_I^{(\varepsilon)}, \quad I \in \mathcal{D}, \quad (5.14)$$

$$B_m^{(\varepsilon)} h_I = b_I^{(\varepsilon)}, \quad I \in \mathcal{D}, \quad (5.15)$$

$$B_m^{(\varepsilon)} h_I = b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}, \quad I \in \mathcal{D}, \quad (5.16)$$

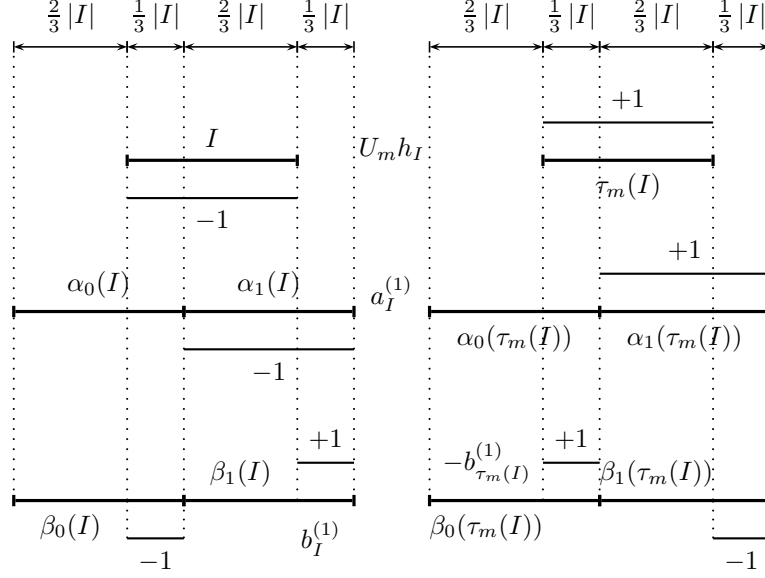
for $\varepsilon \in \{0, 1\}$. Note the identities

$$U_m = A_m^{(\varepsilon)} + B_m^{(\varepsilon)} = A_m^{(\varepsilon)} + B^{(\varepsilon)} - B^{(\varepsilon)} \circ T_m, \quad (5.17)$$

hold true for $\varepsilon \in \{0, 1\}$, see (5.10), (5.11), (5.12), (5.13) and Figures 6 and 7.

Now we split the collections $\mathcal{B}_i^{(1)}$ into

$$\mathcal{B}_i^{(1)} = \mathcal{B}_i^{(1,0)} \cup \mathcal{B}_i^{(1,1)}, \quad (5.18)$$

FIGURE 7. Martingale decomposition of U_m to the right.

where

$$\mathcal{B}_i^{(1,0)} = \{I \in \mathcal{B}_i^{(1)} : \inf \tau_m(I) \neq \inf \pi^\lambda(\tau_m(I))\}, \quad (5.19)$$

$$\mathcal{B}_i^{(1,1)} = \{I \in \mathcal{B}_i^{(1)} : \inf \tau_m(I) = \inf \pi^\lambda(\tau_m(I))\}, \quad (5.20)$$

for all $0 \leq i \leq K(m)$. The projections

$$P_i^{(0)} u = \sum_{I \in \mathcal{B}_i^{(0)}} \langle u, h_I \rangle h_I |I|^{-1}$$

were defined in (4.16), accordingly we set

$$P_i^{(1,\varepsilon)} u = \sum_{I \in \mathcal{B}_i^{(1,\varepsilon)}} \langle u, h_I \rangle h_I |I|^{-1}, \quad (5.21)$$

for all $0 \leq i \leq K(m)$ and $\varepsilon \in \{0, 1\}$. The collection $\mathcal{B}_i^{(0)}$ is specified in Lemma 4.1, and $\mathcal{B}_i^{(1,\varepsilon)}$ is defined in (5.19) and (5.20). Finally, if we define

$$\begin{aligned} P^{(0)} &= \sum_{i=0}^{K(m)} P_i^{(0)}, \\ P^{(1,\varepsilon)} &= \sum_{i=0}^{K(m)} P_i^{(1,\varepsilon)}, \end{aligned} \quad (5.22)$$

for all $\varepsilon \in \{0, 1\}$, then certainly

$$u = P^{(0)} u + P^{(1,0)} u + P^{(1,1)} u \quad (5.23)$$

for all $u \in L_X^p$. Note that $P_i^{(1)} = P_i^{(1,0)} + P_i^{(1,1)}$, where $P_i^{(1)}$ was defined in (4.16).

In the following theorem the operator U_m is decomposed into five parts, each of which is forming a martingale difference sequence.

Theorem 5.1. *Let $m \in \mathbb{Z}$, $m \geq 1$ and $0 \leq i \leq K(m)$. Then the identity*

$$U_m u = U_m \circ P^{(0)} u + \sum_{\varepsilon \in \{0,1\}} (A_m^{(\varepsilon)} + B_m^{(\varepsilon)}) \circ P^{(1,\varepsilon)} u \quad (5.24)$$

holds true for all $u \in L_X^p$. For every $0 \leq i \leq K(m)$ and $\varepsilon \in \{0,1\}$, each of the following collections is a martingale difference sequence:

$$\{U_m \circ P_i^{(0)} h_I : I \in \mathcal{D}\}, \quad (5.25)$$

$$\{A_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} h_I : I \in \mathcal{D}\}, \quad (5.26)$$

$$\{B_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} h_I : I \in \mathcal{D}\}. \quad (5.27)$$

We have the estimate $K(m) \leq 7+2 \cdot \log_2(m)$, where $K(m)$ is defined in Lemma 4.1 on page 8.

Remark 5.2. For reasons of symmetry, a similar result holds true for $m \leq -1$, when adjusting the construction of a_I , b_I and $\mathcal{B}_i^{(1,\varepsilon)}$, accordingly.

Proof. Let $m \in \mathbb{Z}$, $m \geq 1$ and $0 \leq i \leq K(m)$ be fixed throughout the rest of this proof. Whenever we apply the predecessor map π to an interval $I \in \sigma^\delta(\mathcal{D})$, we understand it with respect to $\sigma^\delta(\mathcal{D})$, with $\delta \in \{0,1\}$ fixed.

Observe, identity (5.24) follows immediately from (5.23) and (5.17).

First, note that Lemma 4.1 implies that

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \mathcal{B}_i^{(0)}\}$$

is a nested collection of sets, hence

$$\{U_m h_I : I \in \mathcal{B}_i^{(0)}\}$$

is a martingale difference sequence.

Secondly, we will show that $\{a_I^{(0)} : I \in \mathcal{B}_i^{(1,0)}\}$ forms a martingale difference sequence. Henceforth, we shall abbreviate $\mathcal{B}_i^{(1,0)}$ by \mathcal{B} . Now, fix $I, J \in \mathcal{B}$, $|J| < |I|$ such that $\text{supp } a_J^{(0)} \cap \text{supp } a_I^{(0)} \neq \emptyset$. Note that $J \subset (\pi^\lambda(J))_{11}$, for all $J \in \mathcal{B}$, where K_{11} , $K \in \mathcal{D}$ denotes the unique $M \subset K$, $M \in \mathcal{D}$, $|M| = |K|/4$ such that $\sup M = \sup K$. From this and the definition of \mathcal{B} it is clear that $\text{supp } a_J^{(0)} \subset \alpha_1(\pi^\lambda(J))$ (see also Remark 4.2), hence

$$\emptyset \neq \alpha_1(\pi^\lambda(J)) \cap \text{supp } a_I^{(0)} = (\alpha_1(\pi^\lambda(J)) \cap \alpha_0(I)) \cup (\alpha_1(\pi^\lambda(J)) \cap \alpha_0(\tau_m(I))).$$

Since $|J| < |I|$, $I, J \in \mathcal{B}$, we know that $|\alpha_1(\pi^\lambda(J))| \leq |I|$, thus

$$\text{either } \alpha_1(\pi^\lambda(J)) \subset \alpha_0(I) \quad \text{or} \quad \alpha_1(\pi^\lambda(J)) \subset \alpha_0(\tau_m(I)),$$

which finishes the second part of this proof.

The proof that $\{a_I^{(1)} : I \in \mathcal{B}_i^{(1,1)}\}$ forms a martingale difference sequence is essentially the same, and we omit the details.

Thirdly, we will show that $\{b_I^{(0)}, b_{\tau_m(I)}^{(0)} : I \in \mathcal{B}_i^{(1,0)}\}$ constitutes a martingale difference sequence. Again, we shall abbreviate $\mathcal{B}_i^{(1,0)}$ by \mathcal{B} . To this end, we assume there exist $I, J \in \mathcal{B} \cup \tau_m(\mathcal{B})$, $|J| < |I|$ such that

$$\beta(J) \cap \beta(I) \neq \emptyset \quad \text{and} \quad \beta(J) \cap \beta(I)^c \neq \emptyset. \quad (\mathcal{A})$$

Since $\beta(J) \subset \gamma(J)$, assumption (\mathcal{A}) is covered by the following four cases.

- (1) $\gamma(J) \cap I \neq \emptyset$ and $\gamma(J) \cap I^c \neq \emptyset$,
- (2) $\gamma(J) \cap \gamma_0(I) \neq \emptyset$ and $\gamma(J) \cap \gamma_0(I)^c \neq \emptyset$,
- (3) $\gamma(J) \subset I$ and $\inf \beta_1(I) \in \gamma(J)$,
- (4) $\gamma(J) \subset \gamma_0(I)$ and $\inf \beta_0(I) \in \gamma(J)$.

If we assume case (1), then $\inf J = \inf I$ or $\inf J = \sup I$. Anyhow, we have that $\inf J = \inf \pi^\lambda(J)$, so we know $J \notin (\mathcal{B} \cup \tau_m(\mathcal{B}))$, contradicting our assumption. Case (2) is analogous to case (1). Note that we abbreviated $\mathcal{B}_i^{(1,0)}$ by \mathcal{B} , so consider the definition of $\mathcal{B}_i^{(1)}$ to see that $J \notin \mathcal{B}_i^{(1,0)}$, and consider (5.19) to determine that also $J \notin \tau_m(\mathcal{B}_i^{(1,0)})$.

Let us now assume case (3) is true. This means that either $\inf I + \frac{1}{3}|I| \in \gamma(J)$ or $\inf I + \frac{2}{3}|I| \in \gamma(J)$, depending on the sign of the one-third-shift for I . We fix $z \in \{1, 2\}$ and assume that

$$\inf I + \frac{z}{3}|I| \in \gamma(J). \quad (5.28)$$

Due to (5.19) we see that $\pi^\lambda(\gamma_0(J)) = \pi^\lambda(J)$, so if we set $K = \pi^\lambda(J)$, then

$$\inf I + \frac{z}{3}|I| \in K.$$

This corresponds to either one of the following being true

$$\inf I + \frac{z}{3}|I| = \inf K + \frac{1}{3}|K| \quad \text{or} \quad \inf I + \frac{z}{3}|I| = \inf K + \frac{2}{3}|K|. \quad (5.29)$$

If $J \in \mathcal{B}$ we know $J \subset K_{11}$, thus

$$\begin{aligned} \inf \gamma(J) &\geq \inf K + \frac{3}{4}|K| - 2^{-\lambda}|K| \\ &> \inf K + \frac{2}{3}|K|. \end{aligned} \quad (5.30)$$

Recall that K_{11} denotes the unique $M \subset K$, $M \in \mathcal{D}$, $|M| = |K|/4$ such that $\sup M = \sup K$. The last strict inequality holds true since $\lambda \geq 4$ per construction of \mathcal{B} , see (4.7) if $|m| \geq 2$ and note the exception for $|m| = 1$ beneath. Combining (5.28) and (5.30) yields

$$\inf I + \frac{z}{3}|I| > \inf K + \frac{2}{3}|K|,$$

which contradicts (5.29) in both cases.

If $J \in \tau_m(\mathcal{B})$ we know $J \subset K_{00}$, where K_{00} denotes the unique $M \subset K$, $M \in \mathcal{D}$, $|M| = |K|/4$ such that $\inf M = \inf K$. So we note

$$\begin{aligned} \sup \gamma(J) &\leq \inf K + \frac{1}{4}|K| + 2^{-\lambda}|K| \\ &< \inf K + \frac{1}{3}|K|. \end{aligned} \quad (5.31)$$

The last strict inequality holds true since $\lambda \geq 4$ per construction of \mathcal{B} , see (4.7) if $|m| \geq 2$ and note the exception for $|m| = 1$ beneath. Combining (5.28) and (5.31) yields

$$\inf I + \frac{z}{3}|I| < \inf K + \frac{1}{3}|K|,$$

which contradicts (5.29) in both cases.

Case (4) is analogous to case (3).

Altogether we proved that our assumption (\mathcal{A}) was false, therefore

$$\beta(J) \subset \beta_0(I) \quad \text{or} \quad \beta(J) \subset \beta_1(I)$$

for all $I, J \in \mathcal{B}$, $|J| < |I|$ such that $\beta(J) \cap \beta(I) \neq \emptyset$. In other words, the support of b_J is contained in a set where $b_I^{(0)}$ is constant, hence

$$\{b_I^{(0)}, b_{\tau_m(I)}^{(0)} : I \in \mathcal{B}_i^{(1,0)}\}$$

constitutes a martingale difference sequence.

The proof that $\{b_I^{(1)}, b_{\tau_m(I)}^{(1)} : I \in \mathcal{B}^{(1,1)}\}$ constitutes a martingale difference sequence is essentially the same argument, so we omit it. \square

Remark 5.3. Note in Theorem 5.1 we actually proved the following stronger result. For every $0 \leq i \leq K(m)$ and $\varepsilon \in \{0, 1\}$, the collection

$$\{b_I^{(\varepsilon)}, b_{\tau_m(I)}^{(\varepsilon)} : I \in \mathcal{B}_i^{(1,\varepsilon)}\}$$

is a martingale difference sequence, which certainly implies (5.27).

Consider the splitting of \mathcal{D} into the sets $\mathcal{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0, 1\}$, see Lemma 4.1 on page 8 for details, which we used in Theorem 4.3 on page 10 to treat the shift operator T_m . Retracing our steps in the proof of Theorem 4.3 we find that we could actually repeat this proof with the operator T_m replaced by any of the operators $U_m \circ P^{(0)}$, $A_m^{(\varepsilon)} \circ P^{(1,\varepsilon)}$, $B_m^{(\varepsilon)} \circ P^{(1,\varepsilon)}$, $\varepsilon \in \{0, 1\}$. The details are elaborated in Theorem 5.4 below.

Theorem 5.4. *Let $m \in \mathbb{Z}$ and $m \geq 1$. Then for all $0 \leq i \leq K(m)$ and $\varepsilon \in \{0, 1\}$, we have the estimates*

$$\begin{aligned} \|U_m \circ P_i^{(0)} u\|_{L_X^p} &\leq C \cdot \|P_i^{(0)} u\|_{L_X^p}, \\ \|U_m \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} &\leq C \cdot \|P_i^{(1,\varepsilon)} u\|_{L_X^p}, \end{aligned} \tag{5.32}$$

for all $u \in L_X^p$, where the constant C depends only on $\mathcal{U}_p(X)$. Furthermore, we have the bound $K(m) \leq 7 + 2 \cdot \log_2(m)$.

Remark 5.5. For reasons of symmetry, the same result holds true for $m \leq -1$, that is besides the appropriate modifications for $P_i^{(0)}$ and $P_i^{(1,\varepsilon)}$.

Proof. Let $m \in \mathbb{Z}$, $m \geq 1$ and $0 \leq i \leq K(m)$ be fixed throughout the rest of the proof.

First, we will estimate $U_m \circ P_i^{(0)}$. Due to Theorem 5.1 respectively Remark 5.3 we know that $\{U_m \circ P_i^{(0)} h_I : I \in \mathcal{D}\}$ forms a martingale difference sequence, which enables us to introduce Rademacher functions via the UMD-property. Hence

$$\begin{aligned} \|U_m \circ P_i^{(0)} u\|_{L_X^p} &\approx \int_0^1 \left\| \sum_{I \in \mathcal{B}_i^{(0)}} r_I(t) \langle u, h_I \rangle U_m h_I |I|^{-1} \right\|_{L_X^p} dt \\ &= \int_0^1 \left\| \sum_{I \in \mathcal{B}_i^{(0)}} r_I(t) \langle u, h_I \rangle (\text{Id} + T_m) h_I |I|^{-1} \right\|_{L_X^p} dt \end{aligned}$$

for all $u \in L_X^p$. This is all we need to repeat the proof of Theorem 4.3 in Section 4 with T_m replaced by $\text{Id} + T_m$.

Now we turn to the estimate for $U_m \circ P_i^{(1,\varepsilon)}$, with $\varepsilon \in \{0, 1\}$ fixed throughout the rest of the proof. Observe that

$$U_m \circ P_i^{(1,\varepsilon)} u = A_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u + B_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u,$$

for all $u \in L_X^p$, see (5.17). Theorem 5.1 on page 16 ensures that both

$$\{A_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} h_I : I \in \mathcal{D}\} \quad \text{and} \quad \{B_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} h_I : I \in \mathcal{D}\}$$

form martingale difference sequences, which allows us to introduce Rademacher means via the UMD-property, hence

$$\|A_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} \lesssim \int_0^1 \left\| \sum_{I \in \mathcal{B}_i^{(1,\varepsilon)}} r_I(t) \langle u, h_I \rangle a_I^{(\varepsilon)} |I|^{-1} \right\|_{L_X^p} dt$$

and

$$\|B_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} \lesssim \int_0^1 \left\| \sum_{I \in \mathcal{D}_i^{(1,\varepsilon)}} r_I(t) \langle u, h_I \rangle (b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}) h_I |I|^{-1} \right\|_{L_X^p} dt,$$

for all $u \in L_X^p$. Now we can essentially repeat the proof of Theorem 4.3 in Section 4, for $\delta = 1$ and with T_m replaced by $A_m^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$, respectively. We have to utilize the unilateral operators S_0 and S_1 instead of S as well, see Subsection 3.2 on page 6. If we do so, we end up with the estimates

$$\|A_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} \lesssim \int_0^1 \left\| \sum_{I \in \mathcal{D}_i^{(1,\varepsilon)}} r_I(t) \langle u, h_I \rangle h_{\alpha_\varepsilon(I)} |I|^{-1} \right\|_{L_X^p} dt$$

and

$$\|B_m^{(\varepsilon)} \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} \lesssim \int_0^1 \left\| \sum_{I \in \mathcal{D}_i^{(1,\varepsilon)}} r_I(t) \langle u, h_I \rangle b_I^{(\varepsilon)} |I|^{-1} \right\|_{L_X^p} dt,$$

for all $u \in L_X^p$. Thus, considering $h_{\alpha_\varepsilon(I)} = S_\varepsilon h_I$ and $|b_I^{(\varepsilon)}| \leq |S_0 h_I| + |S_1 h_I|$, see (3.9), (3.10), (3.11), (3.12) and combining our estimates for $A_m^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$ with the inequalities for the unilateral one-third-shift operators S_0 and S_1 in Theorem 3.3 on page 7 yields

$$\begin{aligned} \|U_m \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} &\lesssim \|S_0 \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} + \|S_1 \circ P_i^{(1,\varepsilon)} u\|_{L_X^p} \\ &\lesssim \|P_i^{(1,\varepsilon)} u\|_{L_X^p}, \end{aligned}$$

for all $u \in L_X^p$, concluding the proof. \square

From the results established in Theorem 4.3 on page 10 and Theorem 5.4 on the facing page one can obtain the estimates stated in Theorem 5.6 below, by exploiting the type and cotype inequalities for T_m , and only the cotype inequality for U_m . Inserting Theorem 4.3 on page 10 and Theorem 5.4 on the facing page into [Fig88, Lemma 1] one can obtain [Fig88, Theorem 1] stated below for sake of completeness.

Theorem 5.6 ([Fig88]). *Let $1 < p < \infty$, and X be a Banach space with the UMD-property. For $m \in \mathbb{Z}$ let the map τ_m denote the shift map defined by*

$$I \mapsto I + m |I|.$$

Let T_m, U_m denote the linear extensions of the maps

$$T_m h_I = h_{\tau_m(I)},$$

and

$$U_m h_I = \mathbf{1}_{\tau_m(I)} - \mathbf{1}_I,$$

respectively, then

$$\begin{aligned} \|T_m : L_X^p \rightarrow L_X^p\| &\leq C (\log_2(2 + |m|))^\alpha, \\ \|U_m : L_X^p \rightarrow L_X^p\| &\leq C (\log_2(2 + |m|))^\beta, \end{aligned}$$

where the constant $C > 0$ depends only $\mathcal{U}_p(X)$ and $0 < \alpha, \beta < 1$. Moreover, if L_X^p has type \mathcal{T} and cotype \mathcal{C} , then one can take $\alpha = \frac{1}{\mathcal{T}} - \frac{1}{\mathcal{C}}$ and $\beta = 1 - \frac{1}{\mathcal{C}}$.

REFERENCES

- [Bou86] J. Bourgain. Vector-valued singular integrals and the H^1 -BMO duality. Probability theory and harmonic analysis, Pap. Mini-Conf., Cleveland/Ohio 1983, Pure Appl. Math., Marcel Dekker 98, 1-19 (1986)., 1986.
- [Bur81] D. L. Burkholder. A Geometrical Characterization of Banach Spaces in which Martingale Difference Sequences are Unconditional. *Annals of Probability*, 9(6):997–1011, 1981.
- [CWW85] S.Y.A. Chang, J.M. Wilson, and T.H. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.*, 60:217–246, 1985.
- [Dav80] Burgess Davis. Hardy spaces and rearrangements. *Trans. Amer. Math. Soc.*, 261(1):211–233, 1980.
- [Fig88] T. Figiel. On Equivalence of Some Bases to the Haar System in Spaces of Vector-valued Functions. *Bulletin of the Polish Academy of Sciences*, 36(3–4):119–131, 1988.
- [Fig90] T. Figiel. Singular Integral Operators: A Martingale Approach. In *Geometry of Banach Spaces*, number 158 in London Mathematical Society Lecture Note Series, pages 95–110, 1990.
- [FW01] T. Figiel and P. Wojtaszczyk. Special bases in function spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 561–597. North-Holland, Amsterdam, 2001.
- [GJ82] John B. Garnett and Peter W. Jones. BMO from dyadic BMO. *Pac. J. Math.*, 99:351–371, 1982.
- [Hyt11] T. P. Hytonen. Foundations of vector-valued singular integrals revisited—with random dyadic cubes. <http://arxiv.org/abs/1110.5826>, [v1] Wed, 26 Oct 2011, 2011.
- [Kah85] J.-P. Kahane. *Some Random Series of Functions, Second Edition*. Cambridge University Press, 1985.
- [Lec11] R. Lechner. An Interpolatory Estimate and Shift Operators. *Ph.D. thesis, July 2011*, http://shrimp.bayou.uni-linz.ac.at/Papers/dvi/phd_thesis_Richard_Lechner.pdf, 2011.
- [MP11] Paul F. X. Müller and Markus Passenbrunner. A decomposition theorem for singular integral operators on spaces of homogeneous type. *To appear in the Journal of Functional Analysis*, 2011.
- [Mül05] Paul F. X. Müller. *Isomorphisms between H^1 spaces*. Monografie Matematyczne. Instytut Matematyczny PAN (New Series) 66. Basel: Birkhäuser., 2005.
- [NS97] I. Novikov and E. Semenov. *Haar series and linear operators*. Kluwer Academic Publishers, 1997.
- [NTV97] F. Nazarov, S. Treil, and A. Volberg. Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces. *Internat. Math. Res. Notices*, 15:703–726, 1997.
- [NTV03] F. Nazarov, S. Treil, and A. Volberg. The Tb -theorem on non-homogeneous spaces. *Acta Math.*, 190(2):151–239, 2003.
- [Ste70] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.
- [Wol82] Thomas H. Wolff. Two algebras of bounded functions. *Duke Math. J.*, 49:321–328, 1982.

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